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On the Auxetic Properties of 'Rotating Rectangles' with Different Connectivity

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Auxetic materials and structures exhibit the unusual property of becoming wider when stretched and thinner when compressed, i.e., they have negative Poisson's ratios.[1)] In recent years, this unusual behaviour has been predicted or experimentally measured in a number of naturally-occurring and man-made materials ranging from foams where the auxetic effect arises from the particular microstructure of the foams[2–5)] to silicates and zeolites where the auxetic behaviour occurs at the molecular level.[6–10)] In these auxetic systems, the negative Poisson's ratios can be explained in terms of models based on the geometry of the system (i.e., the geometry of the material's internal structure) and the way this geometry changes as a result of applied loads (deformation mechanism).

In recent years various two and three dimensional theoretical models and structures which can lead to negative Poisson's ratio have been proposed including, two and three-dimensional 're-entrant' systems,[1–3,11–15)] models based on rigid 'free' molecules,[16–18)] chiral structures[4,19–22)] and systems made from 'rotating rigid units' such as squares, triangles, rectangles or tetrahdera.[22–28)] In all of these systems, the Poisson's ratio does not depend on scale although it can depend on the relative dimensions of certain features in the geometry. In particular we have recently shown that whilst a two-dimensional system constructed from perfectly rigid squares connected together through simple hinges at the vertices of the squares will always maintain its aspect ratio when stretched or compressed [see Fig. 1(a)], i.e., it will exhibit constant Poisson's ratios equal to -1 irrespective of the size of the square or direction of loading,[23,24)] the equivalent structure built from hinged rigid rectangles as illustrated in Fig. 1(b) will exhibit in-plane Poisson's ratios which depend on the shape of the rectangles (the ratio of the lengths of the two sides) and the relative orientation of the rectangles (i.e., the angles that two adjacent rectangles make with respect to each other). This means that for such a system, the Poisson's ratios will be strain dependent and dependent on the direction of loading.[24,28)]

This note is aimed at highlighting the fact that there exist two types of 'rotating rectangles' structures, and that two systems based on the same 'building block' (rigid rectangle) and same deformation mechanism (rectangle rotation), but different connectivity, exhibit very different mechanical properties.

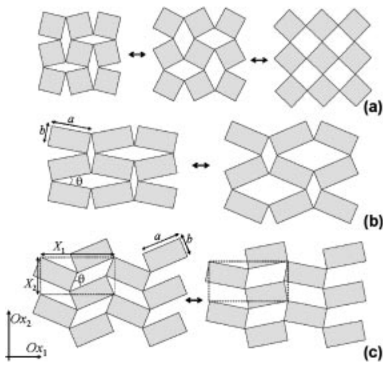

Fig. 1. (a) The 'rotating squares' system, (b) the Type I and (c) Type II 'rotating rectangles' systems.

More specifically, for rectangles of the same size $(a \times b)$, tessellating corner-sharing rectangular networks in which each corner is shared between two rectangles can only be formed from two connectivity schemes, which we shall refer to as Type I and Type II. The Type I network refers to the system where four rectangles are connected in such a way that the empty spaces between the rectangles form rhombi of size $(a \times a)$ and $(b \times b)$ as illustrated in Fig. 1(b). The Type II network refers to the system with a connectivity where the empty spaces between the rectangles form parallelograms of size $(a \times b)$ as illustrated in Fig. 1(c). If the four rectangles are connected in any other way (for example, with the empty spaces between the rectangles forming a 'kite' of side lengths 'a, a, b, b') the resulting unit cannot form a tessellating structure.

The Type I 'rotating rectangles' structure has been extensively studied[24,28,29)] and it has been shown that this structure exhibits properties which are dependent on the shape and size of the rectangles and are strain dependent and anisotropic. In particular it has been shown that such Type I 'rotating rectangles' structures are capable of exhibiting both positive and negative Poisson's ratio where, for example, the on-axis Poisson's ratios are dependent on the ratio of the lengths (a/b) and on the angle θ between the rectangles since:

$$\upsilon_{21} = (\upsilon_{12})^{-1} = \frac{a^2 \sin^2\left(\frac{\theta}{2}\right) - b^2 \cos^2\left(\frac{\theta}{2}\right)}{a^2 \cos^2\left(\frac{\theta}{2}\right) - b^2 \sin^2\left(\frac{\theta}{2}\right)}$$

Here we study, for the first time, the behaviour of the Type II 'rotating rectangles' which as we will show exhibits very different properties.

As illustrated in Fig. 1(c), a rectangular unit cell with cell sides are parallel to the Ox_1 and Ox_2 axis may be used to describe the Type II network. This unit cell contains two $(a \times b)$ rectangles with projections in the Ox_i directions given by:

$$X_1 = 2a \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{and} \quad X_2 = 2b \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

If we assume that the structure deforms solely by relative rotation of the rectangles, then a and b are constants and hence X_i are functions of the single variable θ. Also, as in the earlier derivation for the properties of the Type I structure, we shall assume that the stiffness of the structure (and hence the Young's moduli) is a result of the stiffness of the hinges, that is, a stiffness which opposes changes in the angles θ. In particular, we shall assume that the hinges satisfy the equation $M = K_h(\delta\theta)$ where M is the moment applied to the

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rectangles, $\delta\theta$ is the angular displacement due to M, and K_h is the spring constant for the hinge. The rigidity of the rectangles result in a structure which is geometrically constrained not to shear which results in a value of zero for the five elements of compliance matrix which are associated with shearing. Hence the compliance matrix for this system is of the form:

$$\mathbf{S} = [s_{ij}] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where ν_{ij} represents the Poisson's ratios in the Ox_{ij} plane for loading in the Ox_i direction and E_i represent the Young's moduli for loading in the Ox_i directions. In particular, the Poisson's ratio may be defined by:

$$\nu_{ij} = (\nu_{ji})^{-1} = -\frac{d\varepsilon_j}{d\varepsilon_i} \quad i, j = 1, 2$$

where $d\varepsilon_i$ represent infinitesimally small strains in the Ox_i directions which may be defined in terms of infinitesimally small changes 'dX_i' in the value of 'X_i' by:

$$d\varepsilon_i = \frac{dX_i}{X_i}$$

Since $X_i = X_i(\theta)$, the Poisson's ratios may be written as:

$$\nu_{21} = (\nu_{12})^{-1} = -\frac{d\varepsilon_1}{d\varepsilon_2} = -\frac{dX_1/X_1}{dX_2/X_2} = -\frac{dX_1/d\theta}{dX_2/d\theta} \cdot \frac{X_2}{X_1}$$

where since:

$$\frac{dX_1}{d\theta} = a\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{and} \quad \frac{dX_2}{d\theta} = b\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

the Poisson's ratios simplify to:

$$\nu_{21} = (\nu_{12})^{-1} = -1$$

The on-axis Young's moduli for loading in an Ox_i direction can be derived through the conservation of energy approach. The strain energy per unit volume (U) is related to W, the work done per unit cell due to changes '$d\theta$' in each of the four 'θ' angles contained in the unit cell, through the relationship:

$$U = \frac{1}{V}W$$

where

$$U = \frac{1}{2}E_i(d\varepsilon_i)^2 = \frac{1}{2}E_i\left(\frac{dX_i}{X_i}\right)^2 = \frac{1}{2}E_i\left(\frac{1}{X_i}\cdot\frac{dX_i}{d\theta}\right)^2(d\theta)^2$$
$$W = 4\left[\tfrac{1}{2}K_h(d\theta)^2\right]$$

and V is the volume of the unit cell which is given by $V = X_1X_2$ (assuming a unit thickness in the third direction). The Young's moduli are hence given by:

$$E_i = 4K_h\frac{X_i^2}{X_1X_2}\left(\frac{dX_i}{d\theta}\right)^{-2} \quad i, j = 1, 2$$

which in this case simplify to:

$$E_1 = E_2 = 4K_h\left[ab\cos^2\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right]^{-1} = \frac{8K_h}{ab[1 - \sin(\theta)]}$$

The off-axis mechanical properties obtained from the standard transformation equations[30] show that this idealised system is isotropic, which means that the Poisson's ratio has a constant value of -1 irrespective of the direction of loading. This behaviour of the Type II system is in sharp contrast to that of the Type I system which exhibits anisotropic behaviour. Instead, the Type II system mimics exactly the behaviour of the much simpler 'rotating squares' structure. This is a very interesting feature as it highlights the dependence of the Poisson's ratio on the geometry of the system: slight variations, such as a change in the way that the rectangles are connected together, can lead to drastic changes in the Poisson's ratios.

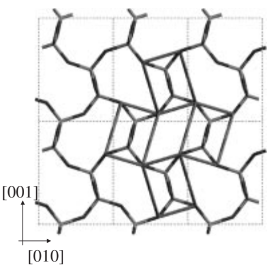

Fig. 2. The (100) plane of the silicate α-cristobalite which can be described in terms of a 'Type II rotating rectangles' geometry (highlighted in the figure). Similar features are also present in the (010) plane.

Finally one should note that for both the Type I and Type II systems, the Poisson's ratios are scale independent and they can be implemented at the micro- or nano-level to produce micro- or nanostructured materials which exhibit auxetic behaviour. In particular, it is noted that when one looks at the (100) and (010) planes of the auxetic silicate α-cristobalite, one may easily identify the Type II 'rotating rectangles' geometry described here (see Fig. 2). In this case, the 'rectangles' represent the two dimensional projection of the silica tetrahedral framework. In such real materials, we would expect deformation of the rectangles to occur in parallel with the rotations, and this could reduce the extent of auxeticity. Whether a material with a 'rotating rectangles' geometry is auxetic or not would depend on which of these two deformation mechanisms dominates, and to what extent it does so.

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